自己相関関数

以下の transfer function $H(s)$ を持つ dynamic system の autocorrelation を計算しテキストの general form と一致することを確認する.

$$H(s) = \frac{as+b}{s^2+2\zeta w_n s + w_n^2} \tag{1}$$

0.1 計算

impulse response $h(t)$ を計算する．$h(t)$ は $H(s)$ の逆ラプラス変換で求めることができる．逆ラプラス変換は

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds = \sum_{\text{all pole of F(s)}} \mathrm{Res}[F(s)e^{st}] \tag{2}$$

を利用して計算する.

$$\begin{aligned} H(s) &= \frac{as+b}{s^2+2\zeta w_n s + w_n^2} &(3)\\ &= \frac{as+b}{(s-\lambda_+)(s-\lambda_-)} &(4)\end{aligned}$$

ここで 2 つの極 (pole) $\lambda_\pm$ は以下のようになり，簡単のため λ_Rとλ_I で表記する.

$$\begin{aligned} \lambda_\pm &= -\zeta w_n \pm i\sqrt{(1-\zeta^2)w_n^2} &(5)\\ &= -\lambda_R \pm i\lambda_I &(6)\end{aligned}$$

このとき $\lambda_R^2+\lambda_I^2 = (\zeta w_n)^2 + (\sqrt{(1-\zeta^2)w_n^2})^2 = w_n^2$ になる.

Residual を計算すると

$$\begin{aligned} \mathrm{Res}_{s=\lambda_+}[H(s)e^{st}] &= \lim_{s\to\lambda_+}(s-\lambda_+)H(s)e^{st} &(7)\\ &= \lim_{s\to\lambda_+}\frac{as+b}{(s-\lambda_-)}e^{st} &(8)\\ &= \frac{a\lambda_+ + b}{(\lambda_+-\lambda_-)}e^{\lambda_+ t} &(9)\\ &= \frac{a(-\lambda_R+i\lambda_I)+b}{2i\lambda_I}\exp((-\lambda_R+i\lambda_I)t) &(10)\\ \mathrm{Res}_{s=\lambda_-}[H(s)e^{st}] &= \lim_{s\to\lambda_-}(s-\lambda_-)H(s)e^{st} &(11)\\ &= \lim_{s\to\lambda_-}\frac{as+b}{(s-\lambda_+)}e^{st} &(12)\\ &= \frac{a\lambda_- + b}{(\lambda_--\lambda_+)}e^{\lambda_- t} &(13)\\ &= -\frac{a(-\lambda_R-i\lambda_I)+b}{2i\lambda_I}\exp((-\lambda_R-i\lambda_I)t) &(14)\end{aligned}$$

よって impulse response $h(t)$ は

$$
\begin{aligned}
h(t) &= \sum_{\text{all pole of H(s)}} \mathrm{Res}[H(s)e^{st}] && (15)\\
&= \mathrm{Res}_{s=\lambda_+}[H(s)e^{st}] + \mathrm{Res}_{s=\lambda_-}[H(s)e^{st}] && (16)\\
&= \frac{a(-\lambda_R + i\lambda_I) + b}{2i\lambda_I}\exp((-\lambda_R + i\lambda_I)t) - \frac{a(-\lambda_R - i\lambda_I) + b}{2i\lambda_I}\exp((-\lambda_R - i\lambda_I)t) && (17)\\
&= \frac{e^{-\lambda_R t}}{2i\lambda_I}\{(a(-\lambda_R + i\lambda_I) + b)e^{i\lambda_I t} - (a(-\lambda_R - i\lambda_I) + b)e^{-i\lambda_I t}\} && (18)\\
&= \frac{e^{-\lambda_R t}}{2i\lambda_I}\{a(-\lambda_R + i\lambda_I)e^{i\lambda_I t} - a(-\lambda_R - i\lambda_I)e^{-i\lambda_I t} + b(e^{i\lambda_I t} - e^{-i\lambda_I t})\} && (19)\\
&= \frac{e^{-\lambda_R t}}{2i\lambda_I}\{-a\lambda_R(e^{i\lambda_I t} - e^{-i\lambda_I t}) + ai\lambda_I(e^{i\lambda_I t} + e^{-i\lambda_I t}) + b(e^{i\lambda_I t} - e^{-i\lambda_I t})\} && (20)\\
&= \frac{e^{-\lambda_R t}}{2i\lambda_I}\{(b - a\lambda_R)(e^{i\lambda_I t} - e^{-i\lambda_I t}) + ai\lambda_I(e^{i\lambda_I t} + e^{-i\lambda_I t})\} && (21)
\end{aligned}
$$

自己相関は入力の分散を D とすると

$$
\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)]\\
&= \mathrm{E}\Big[\int_0^\infty h(s)W(t+\tau-s)ds \int_0^\infty h(s')W(t-s')ds'\Big] && (22)\\
&= \int_0^\infty \int_0^\infty dsds'\ h(s)h(s')\mathrm{E}\Big[W(t+\tau-s)W(t-s')\Big] && (23)\\
&= D\int_0^\infty \int_0^\infty dsds'\ h(s)h(s')\delta(\tau - s + s') && (24)\\
&= D\int_0^\infty ds'\ h(\tau + s')h(s') && (25)\\
&= D\int_0^\infty ds\ h(\tau + s)h(s) && (26)
\end{aligned}
$$

上記で計算した $h(s)$ を代入すると

$$
\begin{aligned}
&= D\int_0^\infty ds\ \frac{e^{-\lambda_R(\tau+s)}}{2i\lambda_I}\{(b-a\lambda_R)(e^{i\lambda_I(\tau+s)}-e^{-i\lambda_I(\tau+s)})+ai\lambda_I(e^{i\lambda_I(\tau+s)}+e^{-i\lambda_I(\tau+s)})\} \\
&\quad\times\frac{e^{-\lambda_R s}}{2i\lambda_I}\{(b-a\lambda_R)(e^{i\lambda_I s}-e^{-i\lambda_I s})+ai\lambda_I(e^{i\lambda_I s}+e^{-i\lambda_I s})\} \quad (27)\\
&= \frac{D}{(2i\lambda_I)^2}\int_0^\infty ds\ e^{-\lambda_R(\tau+s)}\{(b-a\lambda_R)(e^{i\lambda_I(\tau+s)}-e^{-i\lambda_I(\tau+s)})+ai\lambda_I(e^{i\lambda_I(\tau+s)}+e^{-i\lambda_I(\tau+s)})\} \\
&\quad\times e^{-\lambda_R s}\{(b-a\lambda_R)(e^{i\lambda_I s}-e^{-i\lambda_I s})+ai\lambda_I(e^{i\lambda_I s}+e^{-i\lambda_I s})\} \quad (28)\\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\int_0^\infty ds\ e^{-2\lambda_R s}\{(b-a\lambda_R)(e^{i\lambda_I(\tau+s)}-e^{-i\lambda_I(\tau+s)})+ai\lambda_I(e^{i\lambda_I(\tau+s)}+e^{-i\lambda_I(\tau+s)})\} \\
&\quad\times\{(b-a\lambda_R)(e^{i\lambda_I s}-e^{-i\lambda_I s})+ai\lambda_I(e^{i\lambda_I s}+e^{-i\lambda_I s})\} \quad (29)\\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\int_0^\infty ds\ e^{-2\lambda_R s}\{(b-a\lambda_R)^2(e^{i\lambda_I(\tau+2s)}-e^{-i\lambda_I\tau}-e^{i\lambda_I\tau}+e^{-i\lambda_I(\tau+2s)}) \\
&\quad+ai\lambda_I(b-a\lambda_R)(e^{i\lambda_I(\tau+2s)}-e^{-i\lambda_I\tau}+e^{i\lambda_I\tau}-e^{-i\lambda_I(\tau+2s)}) \\
&\quad+ai\lambda_I(b-a\lambda_R)(e^{i\lambda_I(\tau+2s)}+e^{-i\lambda_I\tau}-e^{i\lambda_I\tau}-e^{-i\lambda_I(\tau+2s)}) \\
&\quad+(ai\lambda_I)^2(e^{i\lambda_I(\tau+2s)}+e^{-i\lambda_I\tau}+e^{i\lambda_I\tau}+e^{-i\lambda_I(\tau+2s)})\} \quad (30)\\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\int_0^\infty ds\ e^{-2\lambda_R s}\{(b-a\lambda_R)^2(e^{i\lambda_I(\tau+2s)}-e^{-i\lambda_I\tau}-e^{i\lambda_I\tau}+e^{-i\lambda_I(\tau+2s)}) \\
&\quad+2ai\lambda_I(b-a\lambda_R)(e^{i\lambda_I(\tau+2s)}-e^{-i\lambda_I(\tau+2s)}) \\
&\quad+(ai\lambda_I)^2(e^{i\lambda_I(\tau+2s)}+e^{-i\lambda_I\tau}+e^{i\lambda_I\tau}+e^{-i\lambda_I(\tau+2s)})\} \quad (31)\\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\int_0^\infty ds\ e^{-2\lambda_R s}\{ \\
&\quad((b-a\lambda_R)+ai\lambda_I)^2e^{i\lambda_I(\tau+2s)} \\
&\quad+((b-a\lambda_R)-ai\lambda_I)^2e^{-i\lambda_I(\tau+2s)} \\
&\quad+(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \quad (32)
\end{aligned}
$$

s と関係ない係数を積分の外に出すと，

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\{ \\
&\quad((b-a\lambda_R)+ai\lambda_I)^2 e^{i\lambda_I\tau}\int_0^\infty e^{-2\lambda_R s}e^{i\lambda_I 2s}ds \\
&\quad+((b-a\lambda_R)-ai\lambda_I)^2 e^{-i\lambda_I\tau}\int_0^\infty e^{-2\lambda_R s}e^{-i\lambda_I 2s}ds \\
&\quad+(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\int_0^\infty e^{-2\lambda_R s}ds\} \qquad (33)\\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\{ \\
&\quad((b-a\lambda_R)+ai\lambda_I)^2 e^{i\lambda_I\tau}\int_0^\infty e^{-2(\lambda_R-i\lambda_I)s}ds \\
&\quad+((b-a\lambda_R)-ai\lambda_I)^2 e^{-i\lambda_I\tau}\int_0^\infty e^{-2(\lambda_R+i\lambda_I)s}ds \\
&\quad+(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\int_0^\infty e^{-2\lambda_R s}ds\} \qquad (34)
\end{aligned}$$

λ_R が正であるとすると $\int_0^\infty e^{-As}ds = [\frac{-e^{-As}}{A}]_0^\infty = 1/A \quad (\mathrm{Re}(A)>0)$ と計算できるので

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2}\{ \\
&\quad\frac{((b-a\lambda_R)+ai\lambda_I)^2 e^{i\lambda_I\tau}}{2(\lambda_R-i\lambda_I)} \\
&\quad+\frac{((b-a\lambda_R)-ai\lambda_I)^2 e^{-i\lambda_I\tau}}{2(\lambda_R+i\lambda_I)} \\
&\quad+\frac{(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})}{2\lambda_R}\} \qquad (35)
\end{aligned}$$

各項の係数を整理する．

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2(2(\lambda_R-i\lambda_I)(\lambda_R+i\lambda_I)\lambda_R)}\{ \\
&\quad\lambda_R(\lambda_R+i\lambda_I)((b-a\lambda_R)+ai\lambda_I)^2 e^{i\lambda_I\tau} \\
&\quad+\lambda_R(\lambda_R-i\lambda_I)((b-a\lambda_R)-ai\lambda_I)^2 e^{-i\lambda_I\tau} \\
&\quad+(\lambda_R+i\lambda_I)(\lambda_R-i\lambda_I)(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (36)\\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2(2(\lambda_R^2+\lambda_I^2)\lambda_R)}\{ \\
&\quad\lambda_R(\lambda_R+i\lambda_I)((b-a\lambda_R)+ai\lambda_I)^2 e^{i\lambda_I\tau} \\
&\quad+\lambda_R(\lambda_R-i\lambda_I)((b-a\lambda_R)-ai\lambda_I)^2 e^{-i\lambda_I\tau} \\
&\quad+(\lambda_R^2+\lambda_I^2)(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (37)
\end{aligned}$$

このとき $\lambda_R^2+\lambda_I^2=w_n^2$ を利用しながら整理すると，

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R(\lambda_R+i\lambda_I)((b-a\lambda_R)+ai\lambda_I)((b-a\lambda_R)+ai\lambda_I)e^{i\lambda_I\tau} \\
&+\lambda_R(\lambda_R-i\lambda_I)((b-a\lambda_R)-ai\lambda_I)((b-a\lambda_R)-ai\lambda_I)e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (38)\\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R(\lambda_R+i\lambda_I)(b-a(\lambda_R-i\lambda_I))((b-a\lambda_R)+ai\lambda_I)e^{i\lambda_I\tau} \\
&+\lambda_R(\lambda_R-i\lambda_I)(b-a(\lambda_R+i\lambda_I))((b-a\lambda_R)-ai\lambda_I)e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (39)\\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R(b(\lambda_R+i\lambda_I)-a(\lambda_R-i\lambda_I)(\lambda_R+i\lambda_I))((b-a\lambda_R)+ai\lambda_I)e^{i\lambda_I\tau} \\
&+\lambda_R(b(\lambda_R-i\lambda_I)-a(\lambda_R+i\lambda_I)(\lambda_R-i\lambda_I))((b-a\lambda_R)-ai\lambda_I)e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (40)\\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R(b(\lambda_R+i\lambda_I)-aw_n^2)((b-a\lambda_R)+ai\lambda_I)e^{i\lambda_I\tau} \\
&+\lambda_R(b(\lambda_R-i\lambda_I)-aw_n^2)((b-a\lambda_R)-ai\lambda_I)e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (41)\\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R((b\lambda_R-aw_n^2)+bi\lambda_I)((b-a\lambda_R)+ai\lambda_I)e^{i\lambda_I\tau} \\
&+\lambda_R((b\lambda_R-aw_n^2)-bi\lambda_I)((b-a\lambda_R)-ai\lambda_I)e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (42)\\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R((b\lambda_R-aw_n^2)(b-a\lambda_R)-ab\lambda_I^2+i\lambda_I(b^2-a^2w_n^2))e^{i\lambda_I\tau} \\
&+\lambda_R((b\lambda_R-aw_n^2)(b-a\lambda_R)-ab\lambda_I^2-i\lambda_I(b^2-a^2w_n^2))e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2+(ai\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (43)\\
&=\frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2w_n^2(2\lambda_R)}\{ \\
&\lambda_R((b^2\lambda_R-2abw_n^2+a^2w_n^2\lambda_R)+i\lambda_I(b^2-a^2w_n^2))e^{i\lambda_I\tau} \\
&+\lambda_R((b^2\lambda_R-2abw_n^2+a^2w_n^2\lambda_R)-i\lambda_I(b^2-a^2w_n^2))e^{-i\lambda_I\tau} \\
&+w_n^2(-(b-a\lambda_R)^2-(a\lambda_I)^2)(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (44)
\end{aligned}$$

$K_R=(b^2\lambda_R-2abw_n^2+a^2w_n^2\lambda_R), K_I=\lambda_I(b^2-a^2w_n^2), K_{R2}=-w_n^2((b-a\lambda_R)^2+(a\lambda_I)^2)$ と置くと上記の式は

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2 w_n^2 (2\lambda_R)}\{ \\
&\quad \lambda_R(K_R+iK_I)e^{i\lambda_I\tau} \\
&\quad +\lambda_R(K_R-iK_I)e^{-i\lambda_I\tau} \\
&\quad +K_{R2}(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (45) \\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2 w_n^2 (2\lambda_R)}\{ \\
&\quad \lambda_R K_R(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau}) \\
&\quad +i\lambda_R K_I(e^{i\lambda_I\tau}-e^{-i\lambda_I\tau}) \\
&\quad +K_{R2}(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})\} \qquad (46) \\
&= \frac{De^{-\lambda_R\tau}}{(2i\lambda_I)^2 w_n^2 (2\lambda_R)}\{ \\
&\quad (\lambda_R K_R+K_{R2})(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau}) \\
&\quad +i\lambda_R K_I(e^{i\lambda_I\tau}-e^{-i\lambda_I\tau})\} \qquad (47) \\
&= \frac{De^{-\lambda_R\tau}}{-4\lambda_I^2 w_n^2 \lambda_R}\{ \\
&\quad (\lambda_R K_R+K_{R2})(e^{i\lambda_I\tau}+e^{-i\lambda_I\tau})/2 \\
&\quad -\lambda_R K_I(e^{i\lambda_I\tau}-e^{-i\lambda_I\tau})/2i\} \qquad (48)
\end{aligned}$$

オイラーの公式より $\cos x = \frac{e^{ix}+e^{-ix}}{2}, \sin x = \frac{e^{ix}-e^{-ix}}{2i}$ とすると

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{-4\lambda_I^2 w_n^2 \lambda_R}\{(\lambda_R K_R+K_{R2})\cos(\lambda_I\tau)-\lambda_R K_I\sin(\lambda_I\tau)\} \qquad (49)
\end{aligned}$$

$K_R = (b^2\lambda_R - 2abw_n^2 + a^2w_n^2\lambda_R), K_I = \lambda_I(b^2 - a^2w_n^2), K_{R2} = -w_n^2((b-a\lambda_R)^2 + (a\lambda_I)^2)$ より,

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{-4\lambda_I^2 w_n^2 \lambda_R}\{(\lambda_R K_R+K_{R2})\cos(\lambda_I\tau)-\lambda_R K_I\sin(\lambda_I\tau)\} \qquad (50) \\
&= \frac{De^{-\lambda_R\tau}}{-4\lambda_I^2 w_n^2 \lambda_R} \\
&\quad \{(\lambda_R(b^2\lambda_R-2abw_n^2+a^2w_n^2\lambda_R)-w_n^2((b-a\lambda_R)^2+(a\lambda_I)^2))\cos(\lambda_I\tau) \\
&\quad -\lambda_R\lambda_I(b^2-a^2w_n^2)\sin(\lambda_I\tau)\} \qquad (51) \\
&= \frac{De^{-\lambda_R\tau}}{-4\lambda_I^2 w_n^2 \lambda_R}\{(-\lambda_I^2(b^2+a^2w_n^2))\cos(\lambda_I\tau)-\lambda_R\lambda_I(b^2-a^2w_n^2)\sin(\lambda_I\tau)\} \qquad (52) \\
&= \frac{De^{-\lambda_R\tau}}{-4w_n^2}\{(-\frac{1}{\lambda_R}(b^2+a^2w_n^2))\cos(\lambda_I\tau)-\frac{1}{\lambda_I}(b^2-a^2w_n^2)\sin(\lambda_I\tau)\} \qquad (53) \\
&= \frac{De^{-\lambda_R\tau}}{4w_n^2}\{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))\cos(\lambda_I\tau)+\frac{1}{\lambda_I}(b^2-a^2w_n^2)\sin(\lambda_I\tau)\} \qquad (54)
\end{aligned}$$

ここで三角関数の合成を行う．三角関数の合成は $A\cos x + B\sin x = \sqrt{A^2+B^2}\cos(x-\theta)$. ただし $\cos\theta = \frac{A}{\sqrt{A^2+B^2}}, \sin\theta = \frac{B}{\sqrt{A^2+B^2}}$.

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{De^{-\lambda_R\tau}}{4w_n^2}\{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))\cos(\lambda_I\tau) + \frac{1}{\lambda_I}(b^2-a^2w_n^2)\sin(\lambda_I\tau)\} \quad (55)\\
&= \frac{De^{-\lambda_R\tau}}{4w_n^2}\sqrt{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))^2+(\frac{1}{\lambda_I}(b^2-a^2w_n^2))^2}\cos(\lambda_I\tau-\theta) \quad (56)
\end{aligned}$$

ここで $\cos\theta = \frac{((b^2+a^2w_n^2)/\lambda_R)}{\sqrt{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))^2+(\frac{1}{\lambda_I}(b^2-a^2w_n^2))^2}}$．分母に $\cos\theta$ が出てくるように変形すると

$$\begin{aligned}
&\mathrm{E}[X(t+\tau)X(t)] \\
&= \frac{D((b^2+a^2w_n^2)/\lambda_R)}{4w_n^2((b^2+a^2w_n^2)/\lambda_R)}e^{-\lambda_R\tau}\sqrt{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))^2+(\frac{1}{\lambda_I}(b^2-a^2w_n^2))^2}\cos(\lambda_I\tau-\theta) \quad (57)\\
&= \frac{D((b^2+a^2w_n^2)/\lambda_R)}{4w_n^2}e^{-\lambda_R\tau}\frac{\sqrt{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))^2+(\frac{1}{\lambda_I}(b^2-a^2w_n^2))^2}}{((b^2+a^2w_n^2)/\lambda_R)}\cos(\lambda_I\tau-\theta) \quad (58)\\
&= \frac{D((b^2+a^2w_n^2)/\lambda_R)}{4w_n^2}e^{-\lambda_R\tau}\frac{1}{\cos\theta}\cos(\lambda_I\tau-\theta) \quad (59)\\
&= \frac{D((b^2+a^2w_n^2)/\lambda_R)}{4w_n^2}\frac{e^{-\lambda_R\tau}}{\cos\theta}\cos(\lambda_I\tau-\theta) \quad (60)\\
&= \frac{\sigma^2e^{-\lambda_R\tau}}{\cos\theta}\cos(\lambda_I\tau-\theta) \quad (61)
\end{aligned}$$

ここで $\sigma^2 = \frac{D(b^2+a^2w_n^2)}{4w_n^2\lambda_R}$.
最後に $\lambda_R = \zeta w_n, \lambda_I = \sqrt{(1-\zeta^2)w_n^2}$ を代入すると

$$\mathrm{E}[X(t+\tau)X(t)] = \frac{\sigma^2e^{-\zeta w_n\tau}}{\cos\theta}\cos(\sqrt{(1-\zeta^2)w_n^2}\tau-\theta) \quad (62)$$

このとき $\cos\theta = \frac{(b^2+a^2w_n^2)}{\lambda_R\sqrt{(\frac{1}{\lambda_R}(b^2+a^2w_n^2))^2+(\frac{1}{\lambda_I}(b^2-a^2w_n^2))^2}} = \frac{(b^2+a^2w_n^2)}{\zeta w_n\sqrt{(\frac{1}{\zeta w_n}(b^2+a^2w_n^2))^2+(\frac{1}{\sqrt{(1-\zeta^2)w_n^2}}(b^2-a^2w_n^2))^2}}$
, $\sigma^2 = \frac{D(b^2+a^2w_n^2)}{4w_n^2\lambda_R} = \frac{D(b^2+a^2w_n^2)}{4w_n^2\zeta w_n}$